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Gravity, non-commutative geometry and the Wodzicki residue ★

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Abstract

We derive an action for gravity in the framework of non-commutative geometry by using the Wodzicki residue. We prove that for a Dirac operator D on an n dimensional compact Riemannian manifold with $n \geq 4$, n even, the Wodzicki residue $\mathrm{Res}(D^{-n+2})$ is the integral of the second coefficient of the heat kernel expansion of D^2. We use this result to derive a gravity action for commutative geometry which is the usual Einstein–Hilbert action and we also apply our results to a non-commutative extension which is given by the tensor product of the algebra of smooth functions on a manifold and a finite dimensional matrix algebra. In this case we obtain gravity with a cosmological constant.

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1. Introduction

Although General Relativity is well established as a classical theory of gravitational interaction we still do not know how to describe gravity at distances of order Planck-length, i.e., we do not have a theory of gravity which is compatible with the quantum theory of the other fundamental interactions which are experimentally well understood in the framework of the Standard Model.

Since a considerable amount of effort has been spent on this problem one may draw the conclusion that the mathematical concepts of General Relativity have to be

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changed, or more precisely, our classical geometrical concepts may not be well suited for the description of gravity at small distances. A promising direction seems to be the generalization of geometry to non-commutative geometry [1] which has been used by A. Connes and J. Lott [2] to derive a model for the electroweak interactions. This led to a new interpretation of the Higgs particle as a connection on a discrete space. The geometrical set-up studied in Ref. [2] is a tensor product of a compact Riemannian manifold and a discrete two-point space. This geometry leads to theories with one symmetry breaking scale. A detailed presentation of this approach can be found in Ref. [3]. A generalization to geometries where the discrete space has more than two points was performed in Ref. [4] in order to describe theories with several symmetry breaking scales like GUT-models. There is also an alternative approach to non-commutative geometry, which allows for the same geometrical picture, i.e. smooth manifold $\times$ discrete space, but it is not restricted to compact Riemannian space–time. This approach was developed in Refs. [5–7]. A comparison of both models can be found in Ref. [8].

Now it seems to be only natural to apply the concepts of non-commutative geometry to gravity since it is a theory of space–time geometry. A reformulation of gravity in this language brings us in the position to use the power of non-commutative geometry, which might be of some use to find a consistent quantum theory for gravity.

A first step in this direction was made by Chamseddine et al. [9], who generalized the notion of cotangent space to the case where the geometry is given by a tensor product of a smooth manifold and a discrete two-point space. They studied a vielbein and a connection which are related by generalized Cartan structure equations. This led to gravity coupled to a scalar field.

However, in this article we will follow a different line of approach. We shall use the fact that, as described by A. Connes in Ref. [10], the choice of a K-cycle over an algebra specifies the metric properties of the 'manifold' described by the algebra. More precisely, the (geo)metric structure is encoded in the Dirac operator of a K-cycle [1,10]. It is therefore natural to expect that, in the case of classical geometry, it should be possible to derive the curvature tensors and in particular the Einstein–Hilbert action for gravity – scalar curvature times the volume element – from the data provided by the Dirac operator itself. This observation was at the origin of our work. For dimensionality reasons (curvature being homogeneous to the inverse square of a length), one naturally expects the classical gravity action (in dimension four) to be related to the operator D^{-2}. Known properties of the heat kernel of pseudo-differential operators [11] then suggest naturally to define the action for gravity from the logarithmic divergent part of $\mathrm{tr}(D^{-n+2})$ (n is the dimension of the manifold). This approach can then be promoted to the realm of non-commutative geometry.

Such ideas were expressed by A. Connes in 1993. In particular, the claim that the Wodzicki residue of D^{-n+2} leads to gravity appears in his lecture notes [12]. We came independently to the same conclusion via the attempt to understand the relation between the Dixmier trace to the heat kernel expansion of elliptic operators and by learning from Ref. [13] about the relevance of the Wodzicki Residue in this context. One of the main purposes of the present paper is therefore to show explicitly how to relate the lagrangian for gravity to the logarithmic divergent part of $\mathrm{tr}(D^{-n+2})$. This is achieved by exploiting the heat kernel expansion of elliptic operators, see e.g. Ref. [11]. After we had almost finished this work – see the note at the end of the present paper – we learned of an independent and simultaneous calculation [14] (of independent interest because more direct but also less general) leading, in the case of four dimensions, to the same result as ours.

We start in the next section with a brief introduction to the general concepts of non-commutative geometry. The Dixmier trace as an operator theoretic substitute for integration is introduced in Section 3 and we discuss its relation to the heat kernel expansion. We use this as a motivation to derive a gravity action by selecting the second coefficient of the heat kernel expansion. The proof of our main result, namely that the Wodzicki residue of D^{-n+2}, where D is a Dirac operator on an n dimensional compact Riemannian manifold (n even), picks out the second heat kernel coefficient is presented in Section 4. In Section 5 we apply our result to the usual Dirac operator without and with torsion. In this case we obtain the Einstein–Hilbert action. We also consider a simple extension to non-commutative geometry, which is given by the tensor product of the algebra of smooth functions on a manifold and a finite dimensional matrix algebra. Such algebras are used in model building [2,4] and also in Ref. [9]. For those geometrical set-ups we obtain a gravity action with a cosmological constant but no scalar field coupled to gravity and therefore our result is different from that obtained in Ref. [9]. We end this article with some conclusions in Section 6.

2. Dirac-K-cycles and metric structures

In this section we briefly review some of the main concepts of non-commutative geometry in order to make this article self contained and to fix our notation. For a more comprehensive presentation of this subject we refer to Refs. [1,15].

Let $\mathcal{A}$ be an associative unital algebra. We can construct a bigger algebra $\Omega\mathcal{A}$ out of it by associating to each element $a \in \mathcal{A}$ a symbol δa. $\Omega\mathcal{A}$ is the free algebra generated by the symbols a, δa, $a \in \mathcal{A}$ modulo the relation

$$\delta(ab) = \delta a\, b + a\delta b\,. \tag{2.1}$$

With the definition

$$\delta(a_0\delta a_1 \cdots \delta a_k) := \delta a_0\, \delta a_1 \cdots \delta a_k\,, \tag{2.2}$$

$$\delta(\delta a_1 \cdots \delta a_k) := 0\,, \tag{2.3}$$

$\Omega\mathcal{A}$ becomes a $\mathbb{Z}$-graded differential algebra with the odd differential δ and $\delta^2 = 0$. $\Omega\mathcal{A}$ is called the universal differential envelope of $\mathcal{A}$.

The next element in this formalism is a K-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$, where $\mathcal{H}$ is a Hilbert space such that there is an algebra homomorphism

$$\pi : \mathcal{A} \longrightarrow B(\mathcal{H}), \tag{2.4}$$

where $B(\mathcal{H})$ denotes the algebra of bounded operators acting on $\mathcal{H}$. D is an unbounded self-adjoint operator with compact resolvent such that $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$. It is the triple $(\mathcal{A}, \mathcal{H}, D)$ which contains all geometric information.

We can use D to extend π to an algebra homomorphism of $\Omega\mathcal{A}$ by defining

$$\pi(a_0 \delta a_1 \cdots \delta a_k) := \pi(a_0) [D, \pi(a_1)] \cdots [D, \pi(a_k)]. \tag{2.5}$$

However, in general $\pi(\Omega\mathcal{A})$ fails to be a differential algebra. In order to repair this, one has to divide $\Omega\mathcal{A}$ by the two sided $\mathbb{Z}$-graded differential ideal $\mathcal{J}$ given by

$$\mathcal{J} = \bigoplus_{k \in \mathbb{N}} \mathcal{J}^k, \quad \mathcal{J}^k := (\ker \pi)^k + \delta(\ker \pi)^{k-1}. \tag{2.6}$$

Now we can define the non-commutative generalization of the de Rham algebra, $\Omega_D\mathcal{A}$, as [19]

$$\Omega_D\mathcal{A} := \bigoplus_{k \in \mathbb{N}} \pi(\Omega^k\mathcal{A}) / \pi(\mathcal{J}^k). \tag{2.7}$$

$\Omega_D\mathcal{A}$ is an $\mathbb{Z}$-graded differential algebra, where the differential d is defined by

$$d[\pi(\omega)] := [\pi(\delta\omega)], \quad \omega \in \Omega\mathcal{A}. \tag{2.8}$$

If we take, for example, $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions on a compact Riemannian spin manifold M, $\mathcal{H}$ as the space of square-integrable spin-sections and the Dirac operator $D = \not{\partial}$, then $\Omega_D\mathcal{A}$ is the usual de Rham algebra [1].

A remarkable fact is that the geodesic distance $d(p,q)$ on such a manifold M for any $p, q \in M$ is encoded in the Dirac operator D (the algebra is $C(M)$):

$$d(p,q) = \sup\{|a(p) - a(q)|;\ a \in C(M),\ \|[D,a]\| \leq 1\}. \tag{2.9}$$

No arcs are involved on the right hand side of this relation and therefore Eq. (2.9) can be taken as a definition of geodesic distance which still makes sense in situations where arcs cannot be defined. We use this as a motivation to construct an action which only depends on the choice of the Dirac operator for a K-cycle.

3. Dixmier trace and heat kernel expansion

In order to write down an action in the operator theoretic language we need a functional which replaces integration. For Yang–Mills theory the correct substitution is given by the Dixmier trace [1]. It is the unique extension of the usual trace to the class $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ [16], which is an ideal in the algebra of bounded operators. The elements of this ideal are characterized by the condition that for any $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ the ordered eigenvalues λ_i of $|T|$ satisfy

$$\sup_N \frac{1}{\log N} \sum_{i=0}^{N} \lambda_i < \infty. \tag{3.1}$$

On this ideal the Dixmier trace is defined as

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{i=0}^{N-1} \lambda_i . \tag{3.2}$$

An important result in non-commutative geometry is the trace theorem of A. Connes [13], which states that

$$\mathrm{Tr}_\omega(T) = \lim_{p\to 1+} (p-1)\zeta_T(p) , \quad T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}) , \tag{3.3}$$

with

$$\zeta_T(p) = \mathrm{tr}(T^p) . \tag{3.4}$$

If we now take a K-cycle $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H}$ denotes the space of square-integrable sections of a Clifford module[1] on a compact Riemannian manifold M of dimension n and D is a Dirac operator, i.e. a first order elliptic differential operator, then the action functional for Yang–Mills theory is given by

$$I_{\mathrm{YM}} = \mathrm{Tr}_\omega(\Theta^2 |D|^{-n}) , \tag{3.5}$$

where $\Theta \in \Omega^2_D\mathcal{A}$ is the curvature 2-form. The role of $|D|^{-n} \in \mathcal{L}^{(1,\infty)}$ is to bring Θ^2 into the ideal $\mathcal{L}^{(1,\infty)}$. Positivity, finiteness and covariance are ensured by general properties of the Dixmier trace.

Let us study the trace theorem (3.3) for the Yang–Mills action (3.5) in some more detail. We have (assuming $D = |D|$ for simplicity)

$$I_{\mathrm{YM}} = \lim_{p\to n+} (p-n)\, \mathrm{tr}(\Theta^2 D^{-p}) . \tag{3.6}$$

The trace, which is finite for $p > n$, can be rewritten as

$$(\Theta^2 D^{-p}) = \frac{1}{\Gamma(p/2)} \int_0^\infty dt\, \Theta^2 t^{p/2-1} \exp(-tD^2) . \tag{3.7}$$

For this expression we can now apply the heat kernel expansion [17], i.e., there is a unique formal solution to the heat equation

$$(\partial_t + \tilde{\Delta}_x)\boldsymbol{k}_t(x,y) = 0 \tag{3.8}$$

with $\tilde{\Delta}_x = D_x^2$ such that

$$\int_0^\infty dt\, \Theta^2 t^{p/2-1} \exp(-t\tilde{\Delta}_x)\; g(x)^{1/4} s(x) = \int_0^\infty dt\, \Theta^2 t^{p/2-1} \int d^n y \sqrt{g(y)}\, \boldsymbol{k}_t(x,y) s(y) , \quad s \in \mathcal{H} . \tag{3.9}$$

[1] In the algebraic language $\mathcal{H}$ is a finitely generated projective module over $\mathcal{A}$.

The kernel $\boldsymbol{k}_t$ has the expansion

$$\boldsymbol{k}_t(x,y) = q_t(x,y)\sum_{j=0}^{\infty} t^j \boldsymbol{\Phi}_j(x,y)\,, \tag{3.10}$$

where $q_t(x,y) = g(x)^{1/4}(4\pi t)^{-n/2}\exp(-(x-y)^2/4t)$ denotes the Euclidean heat kernel of flat space ($\sqrt{g}$ denotes the canonical density associated to the Riemannian metric on M) and the $\boldsymbol{\Phi}_j$'s can be computed recursively. One finds [17] that

$$\boldsymbol{\Phi}_0(x,x) = \boldsymbol{id}_{\mathrm{End}(\mathcal{H})}\,,\quad \boldsymbol{\Phi}_1(x,x) = \tfrac{1}{6}\boldsymbol{R} - \boldsymbol{F}\,,\quad \ldots\,, \tag{3.11}$$

where $\boldsymbol{R} = R\cdot \boldsymbol{id}_{\mathrm{End}(\mathcal{H})}$, R is the curvature scalar and $\boldsymbol{F}\in \mathrm{End}(\mathcal{H})$ is determined by $\tilde{\Delta}$, i.e. D.

Now we can represent $\Theta^2 D^{-p}$ by the following expansion:

$$\Theta^2 D_x^{-p} s(x) = (4\pi)^{-n/2}\Theta^2 \sum_{j=0}^{\infty}\int_0^{\infty} dt\, t^{(p-n)/2+j-1} \times \int d^n y\sqrt{g(y)}\boldsymbol{\Phi}_j(x,y)\exp(-d^2(x,y)/4t)s(y)\,. \tag{3.12}$$

If one takes the trace of this expansion and considers the limit $p\to n$ the first term in the expansion becomes singular and contributes to the residue. Therefore Eq. (3.5) can be written as

$$I_{\mathrm{YM}} = \int d^n x\sqrt{g}\,\mathrm{tr}(\Theta^2)\,. \tag{3.13}$$

However, if we would take the limit $p\to n-2$ the second term in the expansion would develop the same singularity as the first term does in the case $p\to n$. If we could pick out this term we would obtain an functional which contains the curvature scalar and hence it would be a good candidate for a gravity action. Of course, the first term is in the limit $p\to n-2$ horribly divergent and therefore the whole procedure is ill defined. Thus we have to use a different tool to extract this coefficient. Fortunately there is a unique extension of the Dixmier trace to a larger class of pseudo-differential operators, the Wodzicki residue [18]. This residue will allow us to compute $\mathrm{Res}(D^{-n+2})$. The Wodzicki residue was introduced in non-commutative geometry by A. Connes in Ref. [13] and used in Ref. [12] to derive a generalized Polyakov action where it was also claimed that $\mathrm{Res}(D^{-n+2})$ leads to gravity. In the following section we will show that for $n\geq 4$, n even, $\mathrm{Res}(D^{-n+2})$ picks out the second coefficient of the heat kernel expansion of D^2.

4. Symbols of an inverse Laplacian

In this section we will prove a relation between the symbols of an inverse generalized Laplacian and some intrinsic geometric quantities. This relation will be used later to build gravity actions out of generalized Dirac operators which have in common, that their square is a generalized Laplacian. First we shall introduce some notation and review briefly a few basic properties of pseudo-differential operators (see also Ref. [11]).

In the following M is a compact n-dimensional Riemannian manifold and $\mathcal{E}$ and $\mathcal{E}'$ are (complex) vector bundles of rank r and s on M. An mth order differential operator $L : \Gamma(M,\mathcal{E}) \to \Gamma(M,\mathcal{E}')$ acting on sections of $\mathcal{E}$ may be written in local coordinates for suitable trivializations of $\mathcal{E}$ as

$$(Lu(x))_i = \sum_{j=1}^{r} \sum_{|\alpha|=0}^{m} (-i)^{|\alpha|} \, a_\alpha^{ij}(x) \, \partial_x^\alpha u_j(x) \quad \forall\, i = 1,\dots,s\,, \tag{4.1}$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_i \in \mathbb{N}_0$ is a multi-index with $|\alpha| = \sum_{i=1}^n \alpha_i$, a^{ij} is a $r \times s$-matrix and $\partial_x^\alpha := \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$. Motivated by the Fourier representation (on $\mathbb{R}^n$)

$$u(x) = (2\pi)^{-n/2} \int_{R^n} d\xi \, e^{i\langle x,\xi\rangle} \, \hat{u}(\xi)\,,$$

$$(Lu)(x) = (2\pi)^{-n/2} \int_{R^n} d\xi \, e^{i\langle x,\xi\rangle} \, \sigma^L(x,\xi) \, \hat{u}(\xi)\,, \tag{4.2}$$

one introduces the symbol $\sigma^L(x,\xi)$ associated to L by

$$\sigma^L(x,\xi) = \sum_{k=0}^{m} \sigma_k^L(x,\xi) := \sum_{|\alpha|=0}^{m} a_\alpha(x)\, \xi^\alpha\,, \tag{4.3}$$

with $\xi = \xi_\mu dx^\mu$ and $(x,\xi) \in T^*M$. The leading term $\sigma_m^L(x,\xi)$ is called the principal symbol of L. A differential operator L is called elliptic, if its principal symbol $\sigma_m^L(x,\xi) \in \Gamma(T^*M, \pi^*\mathrm{End}(\mathcal{E}))$ is invertible over the open set $\{(x,\xi) \,|\, \xi \neq 0\}$. A generalized Laplacian $\tilde{\Delta}$ on $\mathcal{E}$ is a second order elliptic differential operator, such that $\sigma_2^{\tilde{\Delta}} = \|\xi\|^2 \cdot \boldsymbol{id}_{\mathrm{End}(\mathcal{E})}$, or equivalently $\tilde{\Delta}$ is given in any system of local coordinates by the expression

$$\tilde{\Delta} = -\boldsymbol{g}^{\mu\nu} \partial_\mu \partial_\nu + \boldsymbol{B}^\mu \partial_\mu + \boldsymbol{C}\,, \tag{4.4}$$

where $\boldsymbol{g}^{\mu\nu} = g^{\mu\nu} \cdot \boldsymbol{id}_{\mathrm{End}(\mathcal{E})}$ and $\boldsymbol{B}(x), \boldsymbol{C}(x) \in \mathrm{End}(\mathcal{E})$. For later purposes it is important to notice here (see Ref. [17]), that given any generalized Laplacian $\tilde{\Delta}$ on $\mathcal{E}$, there exists a connection $\boldsymbol{\nabla}^{\mathcal{E}}$ on $\mathcal{E}$ and a section $\boldsymbol{F}$ of the bundle $\mathrm{End}(\mathcal{E})$, such that $\tilde{\Delta}$ can be written as

$$\tilde{\Delta} = -g^{\mu\nu}(\boldsymbol{\nabla}_\mu^{\mathcal{E}} \boldsymbol{\nabla}_\nu^{\mathcal{E}} - \Gamma_{\mu\nu}^\rho \boldsymbol{\nabla}_\rho^{\mathcal{E}}) + \boldsymbol{F} := \Delta^\nabla + \boldsymbol{F}\,, \tag{4.5}$$

with $\Gamma_{\mu\nu}^\rho$ the components of the Levi-Civita connection and $\Delta^\nabla u = -\mathrm{tr}(\boldsymbol{\nabla}^{T^*M \otimes \mathcal{E}} \boldsymbol{\nabla}^{\mathcal{E}} u)$ the Laplacian corresponding to the connection $\boldsymbol{\nabla}^{\mathcal{E}}$, where trace denotes contraction with the metric $g \in \Gamma(M, TM \otimes TM)$. From differential operators one comes to pseudo-differential operators (ΨDOs for short) by enlarging the space of symbols. An appropriate symbol-space for our purposes is the space $S^m(U)$ defined as follows: Let $S^m(U)$

($m \in \mathbb{R}$, $U \subset \mathbb{R}^n$) be the space of functions $\sigma(x,\xi)$ with $(x,\xi) \in K \times \mathbb{R}^n$ and K a compact subset of U, which satisfy the condition

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq c_{\alpha\beta}(1+\|\xi\|)^{m-|\beta|} \quad \forall (x,\xi) \in K \times \mathbb{R}^n , \tag{4.6}$$

where $c_{\alpha\beta}$ is a constant. We will consider ΨDOs where the matrix entries of $\sigma(x,\xi)$ belong to $S^m(U)$. Many of the main properties remain true when passing from differential operators to ΨDOs. Any ΨDO P may be defined by a complete symbol which has an asymptotic expansion $\sigma^P(x,\xi) \sim \sum_{k=0}^\infty \sigma^P_{m-k}(x,\xi)$, where now m can be any real number, and the $\sigma_{m-k}(x,\xi)$ are matrices of smooth functions, which are still homogeneous in ξ of degree $(m-k)$. The sign $\sim$ denotes equivalent modulo infinitely smoothing operators. The symbol of the composition of two ΨDOs P_1 and P_2 is given by the 'Leibniz rule'

$$\sigma^{P_1 \circ P_2}(x,\xi) \sim \sum_{|\alpha|=0}^{\infty} (-i)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma^{P_1} \, \partial_x^\alpha \sigma^{P_2} \tag{4.7}$$

with $\alpha! = \alpha_1! \cdots \alpha_n!$. The last fact that we have to state here, is that there exists a unique trace (with certain properties) on the algebra of ΨDOs, called the Wodzicki residue [18]. For a ΨDO P, acting on sections of a vector bundle $\mathcal{E}$ over a compact Riemannian manifold M, this is defined by

$$\operatorname{Res}(P) := \frac{\Gamma(n/2)}{2\,\pi^{n/2}} \int_{S^*M} \operatorname{tr}(\sigma^P_{-n}(x,\xi)) , \tag{4.8}$$

with $S^*M = \{(x,\xi) \in T^*M \mid \|\xi\|^2 = 1\}$ the cosphere bundle on M and $\sigma^P_{-n}(x,\xi)$ is the symbol of order $-n = -\dim M$ not necessarily the principal symbol. The coefficient in front of the integral is the normalization[2] of the volume of S^{n-1}. Now we have the tools to discuss another version of A. Connes trace theorem [13]: For M a compact n dimensional Riemannian manifold and P a ΨDO of order $-n$ acting on sections of a (complex) vector bundle $\mathcal{E}$ on M the following relation holds:

$$TR_\omega(P) = \operatorname{Res}(P) . \tag{4.9}$$

Moreover $\operatorname{Res}(P)$ only depends on the conformal class of the metric.

For a proof we also refer to Ref. [15]. In an important work [18] M. Wodzicki has shown, that the residue is the unique extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}(\Gamma_{L^2}(M,\mathcal{E}))$. We will use this fact in the following theorem, which is the main result of our article, where the relevant ΨDO is not of order $-\dim M$.

Theorem 1. *For M a compact n dimensional ($n \geq 4$, even) Riemannian manifold and $\tilde{\Delta}$ a generalized Laplacian acting on sections of a (complex) vector bundle $\mathcal{E}$ on M the following relation holds:*

[2] Other authors may use different normalizations.

$$\mathrm{Res}(\tilde{\Delta}^{-n/2+1}) = \tfrac{1}{2}(n-1) \int_M d^n x \sqrt{g}\ \mathrm{tr}(\boldsymbol{\Phi}_1(x,x,\tilde{\Delta}))\ , \tag{4.10}$$

where on the right-hand side $\boldsymbol{\Phi}_1(x,x,\tilde{\Delta}) = \frac{1}{6}\boldsymbol{R} - \boldsymbol{F}$ *is the diagonal part of the second coefficient of the heat kernel expansion of* $\tilde{\Delta}$.

Remark 2. One important observation (see M. Wodzicki [18, Prop. 7.11 and Remark 7.13] is that $\int_{S^{n-1}} d\xi\, d^n x\, \sqrt{g}\ \mathrm{tr}(\sigma^P_{-n}(x,\xi))$ is a scalar density, even though symbols which are not principal symbols are in general not covariant geometric quantities.

Proof. The proof will be established in three steps:
(i) Calculation of $\sigma^{\tilde{\Delta}^{-n/2+1}}_{-n}(x,\xi)$ by a parametric construction.
(ii) Integration over the cosphere bundle (using normal coordinates).
(iii) Converting the result into geometric quantities.

(i) Let $\sigma^{\tilde{\Delta}}(x,\xi) := \sigma_2 + \sigma_1 + \sigma_0$ with σ_2 proportional to $\boldsymbol{id}_{\mathrm{End}(\mathcal{E})} =: \mathbf{1}$. Introduce a new ΨDO P by $\sigma^P(x,\xi) = \sigma^P_{-2} := (\sigma_2)^{-1}$. According to the composition rule (4.7) we have

$$\begin{aligned}
\sigma^{\tilde{\Delta}\circ P-1} &\sim \sum_{|\alpha|=0}^{\infty} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial^\alpha_\xi \sigma^{\tilde{\Delta}}\, \partial^\alpha_x \sigma_2^{-1} - \mathbf{1} \\
&= \sum_{k=1}^{2} \sum_{|\alpha|=0}^{k} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial^\alpha_\xi \sigma_{|\alpha|+2-k}\, \partial^\alpha_x \sigma_2^{-1} \\
&:= -r(x,\xi)\ .
\end{aligned} \tag{4.11}$$

With the notation $\sigma^{P_1}\circ\sigma^{P_2} := \sigma^{P_1\circ P_2}$ relation (4.11) leads to $\sigma^{\tilde{\Delta}}\circ(\sigma^P\circ(\mathbf{1}-r)^{-1}) \sim \mathbf{1}$. Using the geometric series in symbol-space (this can be done because r is of order -1) one obtains

$$\sigma^{\tilde{\Delta}^{-1}}(x,\xi) \sim \sigma_2^{-1} \circ \sum_{k=0}^{\infty} r^{\circ k}\ . \tag{4.12}$$

We begin to compute

$$\begin{aligned}
r_{-k}(x,\xi) &= \sum_{|\alpha|=0}^{k} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial^\alpha_\xi \sigma_{|\alpha|+2-k}\, \partial^\alpha_x \sigma_2^{-1}\ , \\
r_{-1}(x,\xi) &= -\sigma_2^{-1}\sigma_1 - i\,\sigma_2^{-2}\, \partial_{\xi_\mu}\sigma_2\, \partial_{x^\mu}\sigma_2\ , \\
r_{-2}(x,\xi) &= -\sigma_2^{-1}\sigma_0 - \sigma_2^{-2}(i\,\partial_{\xi_\mu}\sigma_1\, \partial_{x^\mu}\sigma_2 + \tfrac{1}{2}\, \partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2\, \partial_{x^\mu}\partial_{x^\nu}\sigma_2) \\
&\quad + \sigma_2^{-3}\, \partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2\, \partial_{x^\mu}\sigma_2\, \partial_{x^\nu}\sigma_2\ , \\
r_{-k}(x,\xi) &= 0 \qquad \forall\, k > 2\ .
\end{aligned} \tag{4.13}$$

Again by relation (4.7) we further have

$$\sum_{k=0}^{\infty} r^{\circ k} = \sum_{j=0}^{\infty} s_{-j} \quad \text{with } s_0 = \mathbf{1}, s_{-1} = r_{-1}, s_{-2} = r_{-1}^2 + r_{-2}, \dots . \tag{4.14}$$

From this we can read off the symbol of $\tilde{\Delta}^{-1}$:

$$\sigma^{\tilde{\Delta}^{-1}}(x,\xi) \sim \sum_{l=2}^{\infty} \sigma_{-l}^{\tilde{\Delta}^{-1}} \tag{4.15}$$

with

$$\sigma_{-l}^{\tilde{\Delta}^{-1}}(x,\xi) = \sum_{|\alpha|=0}^{l-2} (-i)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_2^{-1} \, \partial_x^\alpha s_{|\alpha|+2-l} \, .$$

We will only need the first three non-vanishing terms:

$$\begin{aligned} \sigma_{-2}^{\tilde{\Delta}^{-1}}(x,\xi) &= \sigma_2^{-1}, \qquad \sigma_{-3}^{\tilde{\Delta}^{-1}}(x,\xi) = \sigma_2^{-1} r_{-1}, \\ \sigma_{-4}^{\tilde{\Delta}^{-1}}(x,\xi) &= \sigma_2^{-1}(r_{-1}^2 + r_{-2}) + i\sigma_2^{-2} \partial_{\xi_\mu}\sigma_2 \, \partial_{x^\mu} r_{-1} \, . \end{aligned} \tag{4.16}$$

More generally we get

$$\begin{aligned} \sigma^{\tilde{\Delta}^{-m}}(x,\xi) &\sim \sigma^{\tilde{\Delta}^{-m+1}} \circ \sigma^{\tilde{\Delta}^{-1}} \\ &\sim \sum_{|\alpha|=0}^{\infty} (-i)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma^{\tilde{\Delta}^{-m+1}} \, \partial_x^\alpha \sigma^{\tilde{\Delta}^{-1}} := \sum_{l=2m}^{\infty} \sigma_{-l}^{\tilde{\Delta}^{-m}} , \end{aligned} \tag{4.17}$$

with

$$\sigma_{-l}^{\tilde{\Delta}^{-m}}(x,\xi) = \sum_{|\alpha|=0}^{l-2m} \sum_{k=2}^{2+l-|\alpha|-2m} (-i)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{|\alpha|+k-l}^{\tilde{\Delta}^{-m+1}} \, \partial_x^\alpha \sigma_{-k}^{\tilde{\Delta}^{-1}} \, . \tag{4.18}$$

Using this and $\sigma_{-2m}^{\tilde{\Delta}^{-m}} \equiv \sigma_2^{-m}$ we get the recursion relations

$$\begin{aligned} \sigma_{-n}^{\tilde{\Delta}^{-n/2+1}}(x,\xi) &= \sum_{|\alpha|=0}^{2} \sum_{k=2}^{4-|\alpha|} (-i)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{|\alpha|+k-n}^{\tilde{\Delta}^{-1/2+2}} \, \partial_x^\alpha \sigma_{-k}^{\tilde{\Delta}^{-1}} \\ &= \sigma_{2-n}^{\tilde{\Delta}^{-n/2+2}} \sigma_2^{-1} + \sigma_{3-n}^{\tilde{\Delta}^{-n/2+2}} \sigma_{-3}^{\tilde{\Delta}^{-1}} + \sigma_{4-n}^{\tilde{\Delta}^{-n/2+2}} \sigma_{-4}^{\tilde{\Delta}^{-1}} \\ &\quad - i\,\partial_{\xi_\mu} \sigma_{3-n}^{\tilde{\Delta}^{-n/2+2}} \, \partial_{x^\mu} \sigma_2^{-1} - i\,\partial_{\xi_\mu} \sigma_2^{-n/2+2} \, \partial_{x^\mu} \sigma_{-3}^{\tilde{\Delta}^{-n/2+2}} \\ &\quad - \tfrac{1}{2} \partial_{\xi_\mu} \partial_{\xi_\nu} \sigma_2^{-n/2+2} \, \partial_{x^\mu} \partial_{x^\nu} \sigma_2^{-1} \end{aligned} \tag{4.19}$$

and again with relation (4.18)

$$\sigma_{3-n}^{\tilde{\Delta}^{-n/2+2}}(x,\xi) = \sigma_{5-n}^{\tilde{\Delta}^{-n/2+3}} \sigma_2^{-1} + \sigma_2^{-n/2+3} \sigma_{-3}^{\tilde{\Delta}^{-1}} - i\,\partial_{\xi_\mu} \sigma_2^{-n/2+3} \, \partial_{x^\mu} \sigma_2^{-1} \, . \tag{4.20}$$

Now we could proceed in calculating $\mathrm{Res}(\tilde{\Delta}^{-n/2+1})$ in an arbitrary coordinate system, by solving the recursion formulas, but some terms would then become rather clumsy, so it is more convenient to pass to Riemannian normal coordinates at this stage.

(ii) The Taylor expansion of the function $g^{\mu\nu}$ in Riemannian normal coordinates $\mathbf{X}$ around a point x_0 up to order $\mathbf{X}^3$ reads :

$$g^{\mu\nu}(\mathbf{X}) = \delta^{\mu\nu} - \tfrac{1}{3} R^{\mu}{}_{\rho}{}^{\nu}{}_{\lambda}(x_0)\, \mathbf{X}^{\rho}\mathbf{X}^{\lambda} + O(\mathbf{X}^3)\,. \tag{4.21}$$

From now on we will calculate everything with respect to these coordinates:

$$\begin{aligned} &r_{-1}(x_0,\xi) = -\sigma_2^{-1}\,\sigma_1\,, \qquad r_{-2}(x_0,\xi) = -\sigma_2^{-1}\,\sigma_0 + \tfrac{2}{3}\,\sigma_2^{-2}\delta^{\rho\sigma} R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\,\xi_\mu\xi_\nu\,,\\ &\sigma_{-2}^{\tilde{\Delta}^{-1}}(x_0,\xi) = \sigma_2^{-1}\,, \qquad \sigma_{-3}^{\tilde{\Delta}^{-1}}(x_0,\xi) = -\sigma_2^{-2}\,\sigma_1\,,\\ &\sigma_{-4}^{\tilde{\Delta}^{-1}}(x_0,\xi) = -\sigma_2^{-2}\,\sigma_0 + \sigma_2^{-3}\left(\sigma_1^2 - 2i\,\delta^{\rho\mu}\,\partial_{x^\rho}\sigma_1\,\xi_\mu + \tfrac{2}{3}\,\delta^{\rho\sigma} R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\,\xi_\mu\xi_\nu\right)\,. \end{aligned} \tag{4.22}$$

With this we can easily solve the recursion relation (4.20)

$$\begin{aligned} &\sigma_{3-n}^{\tilde{\Delta}^{-n/2+2}}(x_0,\xi) = \sigma_{5-n}^{\tilde{\Delta}^{-n/2+3}}\,\sigma_2^{-1} + \sigma_2^{-n/2+3}\,\sigma_{-3}^{\tilde{\Delta}^{-1}}\\ &\quad\leadsto\quad \sigma_{3-n}^{\tilde{\Delta}^{-n/2+2}}(x_0,\xi) = (\tfrac{1}{2}n-2)\,\sigma^{-n/2+2}\,r_{-1}\,. \end{aligned} \tag{4.23}$$

Inserting Eq. (4.23) in relation (4.19) yields

$$\begin{aligned} &\sigma_{-n}^{\tilde{\Delta}^{-n/2+1}}(x_0,\xi) = \sigma_{2-n}^{\tilde{\Delta}^{-n/2+2}}\,\sigma_2^{-1} + (\tfrac{1}{2}n-1)\,\sigma_2^{-n/2+2}\sigma_{-4}^{\tilde{\Delta}^{-1}} + (\tfrac{1}{2}n-2)\,\sigma_2^{-n/2}\sigma_0\\ &\leadsto\quad \sigma_{-n}^{\tilde{\Delta}^{-n/2+1}}(x_0,\xi) = \tfrac{1}{8}(n-2)\left(n\,\sigma_2^{-n/2+2}\sigma_{-4}^{\tilde{\Delta}^{-1}} + (n-4)\,\sigma_2^{-n/2}\,\sigma_0\right)\,. \end{aligned} \tag{4.24}$$

With the help of the following identity:

$$\int_{S^{n-1}} d\xi\; A^{\mu\nu}\,\xi_\mu\xi_\nu = \frac{2\,\pi^{n/2}}{n\,\Gamma(n/2)}\; g_{\mu\nu}\,A^{\mu\nu}\,, \tag{4.25}$$

and using the explicit symbol $\sigma^{\tilde{\Delta}} = g^{\mu\nu}\,\xi_\mu\xi_\nu + i\,\boldsymbol{B}^\mu\xi_\mu + \boldsymbol{C}$ (see Eq. (4.4)) we get

$$\begin{aligned} \int_{S^{n-1}} d\xi\; \sigma_{-4}^{\tilde{\Delta}^{-1}}(x_0,\xi) = &\frac{2\,\pi^{n/2}}{(\frac{1}{2}n-1)!}\\ &\times\left(-\boldsymbol{C} + \frac{4}{n}(\tfrac{1}{6}\,\boldsymbol{R} + \tfrac{1}{2}\,\partial_\mu \boldsymbol{B}^\mu|_{X=0} - \tfrac{1}{4}\,\boldsymbol{B}_\mu\boldsymbol{B}^\mu)\right)\,, \end{aligned} \tag{4.26}$$

and therefore

$$\begin{aligned} \int_{S^{n-1}} d\xi\; \sigma_{-n}^{\tilde{\Delta}^{-n/2+1}}(x_0,\xi) = &\frac{2\,\pi^{n/2}}{(\frac{1}{2}n-2)!}\\ &\times\left(\tfrac{1}{6}\,\boldsymbol{R} - \boldsymbol{C} + \tfrac{1}{2}\,\partial_{x^\mu}\boldsymbol{B}^\mu|_{X=0} - \tfrac{1}{4}\,\boldsymbol{B}_\mu\boldsymbol{B}^\mu\right)\,. \end{aligned} \tag{4.27}$$

(iii) Comparison of Eq. (4.4) with Eq. (4.5), together with the definition $\nabla_\mu^{\mathcal{E}} := \partial_\mu + \boldsymbol{\Omega}_\mu$ leads to

$$\begin{aligned} &\boldsymbol{B}^\mu = \boldsymbol{g}^{\rho\nu}\Gamma^{\mu}_{\rho\nu} - 2\,g^{\mu\nu}\,\boldsymbol{\Omega}_\nu\,,\\ &\boldsymbol{C} = -g^{\mu\nu}(\partial_\mu\boldsymbol{\Omega}_\nu - \Gamma^{\rho}_{\mu\nu}\boldsymbol{\Omega}_\rho + \boldsymbol{\Omega}_\mu\boldsymbol{\Omega}_\nu) + \boldsymbol{F}\,. \end{aligned} \tag{4.28}$$

Passing to normal coordinates we find

$$F(x_0) = C - \tfrac{1}{2}\,\partial_\mu B^\mu|_{x=0} + \tfrac{1}{4} B_\mu B^\mu\ , \tag{4.29}$$

and so we finally arrive at

$$\int_{S^{n-1}} d\xi\ \sigma_{-n}^{\tilde{\Delta}^{-n/2+1}}(x_0,\xi) = \frac{2\,\pi^{n/2}}{(\frac{1}{2}n-2)!}\left(\tfrac{1}{6}\,R - F\right)\ . \tag{4.30}$$

However, as already mentioned above in the remark, the left-hand side of this equation is a scalar density and therefore, because the right-hand side is a covariant quantity, Eq. (4.30) holds in any system of local coordinates. □

With this theorem we now have a tool to construct gravity actions just by choosing a Dirac operator D, squaring it and reading off $F(D^2)$. This will be the topic of the next section.

5. Dirac operators and gravity actions

In this section we will consider three different types of Dirac operators and the gravity actions associated to them via the main theorem of Section 4. For gravity actions on manifolds satisfying the same hypotheses as in the theorem we take

$$I_{\mathrm{GR}} := -\frac{2}{r\,(n-1)}\,\mathrm{Res}(D^{-n+2}) = \int_M d^n x\,\sqrt{g}\,\left(-\tfrac{1}{6}\,R + \tfrac{1}{r}\,\mathrm{tr}(F)\right)\ , \tag{5.1}$$

with r the rank of the involved vector bundle $\mathcal{E}$ and R the scalar curvature of M. By a Dirac operator D we understand an odd first-order elliptic differential operator acting on sections of a $\mathbb{Z}_2$-graded vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$

$$D : \Gamma(M,\mathcal{E}^\pm) \to \Gamma(M,\mathcal{E}^\mp), \tag{5.2}$$

such that D^2 is a generalized Laplacian. For these vector bundles we take Clifford modules over an even-dimensional Riemannian manifold M. Such a Clifford module is a $\mathbb{Z}_2$-graded bundle $\mathcal{E}$ on M with a graded action ('Clifford-action') of the Clifford bundle $\mathcal{CL}(M) = \mathcal{CL}^+ \oplus \mathcal{CL}^-$ on it:

$$\mathcal{CL}^+ \cdot \mathcal{E}^\pm \subset \mathcal{E}^\pm\ , \qquad \mathcal{CL}^- \cdot \mathcal{E}^\pm \subset \mathcal{E}^\mp\ . \tag{5.3}$$

With this data the Dirac-K-cycle reads $(\mathcal{A},\mathcal{H},D) \equiv (C^\infty(M),\Gamma_{L^2}(M,\mathcal{E}),D)$. Probably the easiest example of this setting is the Dirac operator $D = d + d^*$ (called the signature operator if $\dim M$ is divisible by four) acting on the exterior bundle ΛT^*M, here considered as a Clifford module. To this Dirac operator corresponds the Laplace–Beltrami operator $\Delta = D^2 = dd^* + d^*d$, which is the canonical Laplacian associated to ΛT^*M with $F = 0$. For this operator the action is easily calculated to be $I_{\mathrm{GR}} = -\frac{1}{6}\int_M d^n x\,\sqrt{g}\,R$. Another quite important Clifford module is the spinor bundle over a spin manifold. This is the relevant example if one wants to have fermions in the respective physical model. Actually it is known, that for an even-dimensional Riemannian spin manifold any Clifford module is a twisted bundle $\mathcal{E} = \mathcal{W} \otimes \mathcal{S}$, where $\mathcal{W}$ is a bundle on which the $C\mathcal{L}$-action is trivial and $\mathcal{S}$ is the spinor bundle. Before going to explicit examples we should mention our conventions for the representation matrices γ^a of the Euclidean Clifford algebra:

$$\{\gamma^a, \gamma^b\} = -2\,\delta^{ab}\,\mathbf{1}\,, \quad [\gamma^a, \gamma^b] = 2\,\gamma^{ab}\,, \quad \operatorname{tr}\gamma^a = 0 \rightsquigarrow \operatorname{tr}\gamma^{ab} = 0\,. \tag{5.4}$$

Further we should remark, that the coordinate base $\{\gamma^{\mu}(x)\}$ can as usual be converted via the vielbeins $\varepsilon^a_\mu(x)$ to an orthonormal basis $\{\gamma^a\}$ by $\gamma^\mu = \varepsilon^\mu_a\,\gamma^a$.

5.1. Dirac operator of a Clifford connection

Let $\nabla^{\mathcal{E}}$ be a Clifford connection on a Clifford module $\mathcal{E}$ over a compact n (n as above) dimensional Riemannian manifold M. Such a connection is defined (with respect to local coordinates) by the relation

$$[\nabla^{\mathcal{E}}_\mu, \gamma^\nu] = -\gamma^\rho\,\Gamma^\nu_{\rho\mu}\,, \tag{5.5}$$

so that in the most general case $\nabla^{\mathcal{E}}_\mu$ may explicitly be written as

$$\nabla^{\mathcal{E}}_\mu := \boldsymbol{\omega}_\mu + \boldsymbol{A}_\mu\,, \tag{5.6}$$

with $\boldsymbol{\omega}_\mu := \partial_\mu - \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab}$ the Levi-Civita spin connection and $\boldsymbol{A}_\mu = A_\mu \cdot \mathbf{1}$. The corresponding Dirac operator D_∇ associated to $\nabla^{\mathcal{E}}$ reads

$$D_\nabla = \gamma^\mu\,\nabla^{\mathcal{E}}_\mu = \gamma^a \varepsilon^\mu_a\,\nabla^{\mathcal{E}}_\mu\,. \tag{5.7}$$

With the help of the identity

$$\gamma^{\mu\nu}\,[\boldsymbol{\omega}_\mu, \boldsymbol{\omega}_\nu] = \tfrac{1}{2}\,\boldsymbol{R} \tag{5.8}$$

and (5.4), (5.5), we get for D^2_∇ the Lichnerowicz formula

$$\begin{aligned}
D^2_\nabla &= \tfrac{1}{2}\,\{\gamma^\mu\,\nabla^{\mathcal{E}}_\mu, \gamma^\nu\,\nabla^{\mathcal{E}}_\nu\} \\
&= \tfrac{1}{2}\,\{\gamma^\mu, \gamma^\nu\}\,\nabla^{\mathcal{E}}_\mu\nabla^{\mathcal{E}}_\nu + \gamma^\mu\,[\nabla^{\mathcal{E}}_\mu, \gamma^\nu]\,\nabla^{\mathcal{E}}_\nu + \tfrac{1}{2}\,\gamma^\mu\gamma^\nu\,[\nabla^{\mathcal{E}}_\mu, \nabla^{\mathcal{E}}_\nu] \\
&= -g^{\mu\nu}\,\nabla^{\mathcal{E}}_\mu\nabla^{\mathcal{E}}_\nu - \gamma^\mu\gamma^\rho\,\Gamma^\nu_{\mu\rho}\,\nabla^{\mathcal{E}}_\nu + \tfrac{1}{2}\,\gamma^{\mu\nu}\,[\nabla^{\mathcal{E}}_\mu, \nabla^{\mathcal{E}}_\nu] \\
&= -g^{\mu\nu}\,(\nabla^{\mathcal{E}}_\mu\nabla^{\mathcal{E}}_\nu - \Gamma^\rho_{\mu\nu}\,\nabla^{\mathcal{E}}_\rho) + \tfrac{1}{2}\,\gamma^{\mu\nu}\,[\boldsymbol{\omega}_\mu, \boldsymbol{\omega}_\nu] + \gamma^{\mu\nu}\,\partial_\mu A_\nu \\
&= \Delta^\nabla + \tfrac{1}{4}\,\boldsymbol{R} + \boldsymbol{F}^{\mathcal{E}/S}\,,
\end{aligned} \tag{5.9}$$

with Δ^∇ the Laplacian associated to $\nabla^{\mathcal{E}}$ and $\boldsymbol{F}^{\mathcal{E}/S} = \gamma^{\mu\nu}\,\partial_\mu A_\nu$ the so-called twisting curvature (see also Ref. [17]). So we identify for our first example $\boldsymbol{F} = \frac{1}{4}\boldsymbol{R} + \boldsymbol{F}^{\mathcal{E}/S}$ and the resulting gravity action (see (5.1)) reads

$$I_{\mathrm{GR}} = -\frac{1}{r}\int\limits_M d^n x\,\sqrt{g}\,\operatorname{tr}\left(\tfrac{1}{6}\,\boldsymbol{R} - \tfrac{1}{4}\,\boldsymbol{R} - \boldsymbol{F}^{\mathcal{E}/S}\right) = \tfrac{1}{12}\int\limits_M d^n x\,\sqrt{g}\,R\,. \tag{5.10}$$

We recognize that this action is proportional to the usual Einstein–Hilbert action and the A_μ-part, i.e. the gauge field, drops out completely. Actually any connection associated to an additional internal symmetry does not contribute to the action because the relevant terms are of the same type as the ones for the twisting curvature and the respective traces factorize.

5.2. Dirac operator with torsion

Let now $D_{\tilde{\nabla}}$ be the Dirac operator defined by the same data as in the first example plus an additional torsion term, i.e.

$$\tilde{\nabla}^{\mathcal{E}}_\mu := \nabla^{\mathcal{E}}_\mu + T_\mu \,, \tag{5.11}$$

with $T_\mu := \frac{1}{4}\, t_{\mu ab}\, \gamma^{ab} = \frac{1}{4}\, \varepsilon^c_\mu\, t_{cab}\, \gamma^{ab}$ and t_{cab} totally antisymmetric. A calculation analogous to the one above shows

$$\begin{aligned} D^2_{\tilde{\nabla}} &= -g^{\mu\nu}\, \tilde{\nabla}^{\mathcal{E}}_\mu \tilde{\nabla}^{\mathcal{E}}_\nu + \gamma^\mu\, [\tilde{\nabla}^{\mathcal{E}}_\mu, \gamma^\nu]\, \tilde{\nabla}^{\mathcal{E}}_\nu + \tfrac{1}{2}\gamma^\mu\gamma^\nu\, [\tilde{\nabla}^{\mathcal{E}}_\mu, \tilde{\nabla}^{\mathcal{E}}_\nu] \\ &= -g^{\mu\nu}\left(\nabla^{\mathcal{E}}_\mu\nabla^{\mathcal{E}}_\nu - \Gamma^\rho_{\mu\nu}\,(\nabla^{\mathcal{E}}_\rho + T_\rho) + 6\,T_\mu\nabla^{\mathcal{E}}_\nu + [\nabla^{\mathcal{E}}_\mu, T_\nu] + 5\,T_\mu T_\nu\right) \\ &\quad + \tfrac{1}{2}\,\gamma^{\mu\nu}\, [\nabla^{\mathcal{E}}_\mu, \nabla^{\mathcal{E}}_\nu] + \gamma^{\mu\nu}\, [\nabla^{\mathcal{E}}_\mu + \tfrac{1}{2}T_\mu, T_\nu] \,. \end{aligned} \tag{5.12}$$

By introducing the connection $\hat{\nabla}^{\mathcal{E}}_\mu := \nabla^{\mathcal{E}}_\mu + 3\,T_\mu$ and using (5.8) this may be rewritten as

$$\begin{aligned} D^2_{\tilde{\nabla}} &= \Delta^{\hat{\nabla}} + \tfrac{1}{4}\,R + F^{\mathcal{E}/S} + 2\,g^{\mu\nu}\left([\nabla^{\mathcal{E}}_\mu, T_\nu] - \Gamma^\rho_{\mu\nu}\, T_\rho + 2\,T_\mu T_\nu\right) \\ &\quad + \gamma^{\mu\nu}\, [\nabla^{\mathcal{E}}_\mu + \tfrac{1}{2}T_\mu, T_\nu] \,. \end{aligned} \tag{5.13}$$

In this case we find

$$\begin{aligned} \mathrm{tr}(F) &= \mathrm{tr}\Big(\tfrac{1}{4}\,R + F^{\mathcal{E}/S} + 2\,g^{\mu\nu}([\nabla^{\mathcal{E}}_\mu, T_\nu] - \Gamma^\rho_{\mu\nu}\, T_\rho + 2\,T_\mu T_\nu) \\ &\quad + \gamma^{\mu\nu}\, [\nabla^{\mathcal{E}}_\mu + \tfrac{1}{2}T_\mu, T_\nu]\Big) \\ &= \tfrac{1}{4} r\, R + \mathrm{tr}\left(4\,g^{\mu\nu}\, T_\mu T_\nu + \tfrac{1}{2}\,\gamma^{\mu\nu}\, [T_\mu, T_\nu] + \gamma^{\mu\nu}[\omega_\mu, T_\nu]\right) \,. \end{aligned} \tag{5.14}$$

Using the torsion constraints for the Levi-Civita spin connection ω_μ one can easily show, that $\gamma^{\mu\nu}[\omega_\mu, T_\nu]$ is a boundary term, which does not contribute to the action. For the action we thus get

$$I_{\mathrm{GR}} = \int_M d^n x\, \sqrt{g}\, \left(\tfrac{1}{12}\, R - \tfrac{3}{4}\, t_{abc}\, t^{abc}\right) \,. \tag{5.15}$$

The remaining torsion terms also drop out by their equations of motion and we get the same result as in the first example. From the derivation it is clear, that this would also hold if we would start with a connection which is only compatible with the metric, since the functional Res singles out the Levi-Civita spin connection.

5.3. Dirac operator for 'product K-cycles'

As a special case for a product K-cycle we shall now consider the generalized Dirac operator associated to a 'non-commutative two-point space'. Such spaces where used for a derivation of models of the electroweak interactions in non-commutative geometry (see Ref. [2] for example). A similar setting in the context of gravity was studied in Ref. [9]. Such a two-point space is there given by a four dimensional compact Riemannian spin manifold N, where N is supposed to be also a principal-G-bundle (here $G = \mathbb{Z}_2$) $\pi : N \to M$, over a manifold M. Now the relevant Clifford module on which the Dirac operator acts is $\mathcal{E} = \pi_* \mathcal{S}$ a bundle over M with fiber $S_y = \bigoplus_{\pi(x)=y} S_x$ and S_x the fiber of the spinor bundle at $x \in N$. In this case the $\mathbb{Z}_2$-equivariant Dirac operator and its square have the form

$$D = \begin{pmatrix} \gamma^\mu \tilde{\nabla}^{\mathcal{E}}_\mu & \gamma^5 \phi \\ \gamma^5 \phi & \gamma^\mu \tilde{\nabla}^{\mathcal{E}}_\mu \end{pmatrix} \quad \rightsquigarrow \quad D^2 = \begin{pmatrix} D^2_{\tilde{\nabla}} + \phi^2 \mathbf{1} & \gamma^\mu \gamma^5 \partial_\mu \phi \\ \gamma^\mu \gamma^5 \partial_\mu \phi & D^2_{\tilde{\nabla}} + \phi^2 \mathbf{1} \end{pmatrix} \tag{5.16}$$

with $\gamma^5 := \frac{1}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d$. From this and with the knowledge of the preceding examples we can read off the resulting gravity action (with the torsion T already eliminated by its equations of motion):

$$I_{\mathrm{GR}} = \int_M d^4x \sqrt{g} \, \left(\tfrac{1}{12} R + \phi^2 \right) . \tag{5.17}$$

Until now we did not say anything about the nature of ϕ. In Ref. [9] the derivation of a gravity action led to a kinetic term for ϕ, i.e. ϕ is a scalar field. However, in our case there is no kinetic term for ϕ and therefore we interpret the ϕ^2 term as a cosmological constant. Thus our result is different from that obtained in Ref. [9]. However, the authors of Ref. [9] followed a different philosophy, which involves the generalized de Rham algebra $\Omega_D \mathcal{A}$ whereas in our approach only the Dirac operator of a K-cycle is used to derive a gravity action.

Now we will show that this result remains true for product K-cycles over algebras $\mathcal{A}$ which are a tensor product of the algebra of functions $C^\infty(M)$ on M and a finite dimensional unital matrix algebra $\mathcal{A}_{\mathrm{Mat}}$ which in general is a direct sum of matrix algebras. The number of terms in this sum corresponds to the number of points of the discrete geometry. The K-cycle over $C^\infty(M)$ is given by $(D, \mathcal{H}_1)$, where D denotes the usual Dirac operator and $\mathcal{H}_1$ is the Hilbert-space of square-integrable spin-sections. The second K-cycle, the K-cycle over $\mathcal{A}_{\mathrm{Mat}}$, is the tuple $(\mathcal{M}, \mathcal{H}_2)$, where $\mathcal{H}_2$ is a finite dimensional representation space of $\mathcal{A}_{\mathrm{Mat}}$, i.e. $\mathcal{A}_{\mathrm{Mat}} \subset \mathrm{End}(\mathcal{H}_2)$. The Dirac operator for this finite dimensional algebra is given by $\mathcal{M} \in \mathrm{End}(\mathcal{H}_2)$ with [19]

$$\mathrm{tr}(\mathcal{M} a) = 0 , \quad \forall a \in \mathcal{A} . \tag{5.18}$$

The product K-cycle [1] over the product algebra $\mathcal{A}$ is the tuple $(\mathcal{D}, \mathcal{H})$ with

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$
$$\mathcal{D} = D \otimes \mathbf{1} + \gamma^5 \otimes \mathcal{M} . \tag{5.19}$$

One of the main ideas in Ref. [9] was that the geometry of discrete space may depend on points of the manifold M. This means in our case that we allow for a space dependent $\mathcal{M}$. Formally the Dirac operator $\mathcal{D}$ is now a first-order differential operator on $\mathcal{H}' = \Gamma(M,\mathcal{E})$, where $\mathcal{E}$ is a Clifford module with typical fiber $S \otimes \mathcal{H}_2$, S denotes the spinor space. The next step is to take the square of $\mathcal{D}$ and we get

$$\mathcal{D}^2 = D^2 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{M}^2 + \gamma^5 \otimes [\not{\partial}, \mathcal{M}] , \tag{5.20}$$

which again is a generalized Laplacian such that we can apply the theorem of Section 4. Taking into account condition (5.18) the gravity action for this system is given by

$$I'_{\mathrm{GR}} = \int_M d^4x \sqrt{g} \left(\tfrac{1}{12} R + \frac{1}{\dim \mathcal{H}_2} \mathrm{tr}(\mathcal{M}^2) \right) . \tag{5.21}$$

We note that also in this more general 'n-point' case there is no kinetic term for fields contained in the matrix derivative. The reason is that derivatives of $\mathcal{M}$, which could lead to a kinetic term in the Lagrangian, appear only in the third term of Eq. (5.20) and therefore cannot contribute to the Lagrangian because of Eq. (5.18). As for the two-point case we interpret the term $\mathrm{tr}(\mathcal{M}^2)$ in Eq. (5.21) as a cosmological constant. Another consequence of the vanishing of the kinetic term for $\mathcal{M}$ is that the product K-cycle (5.19) is sufficient to describe gravity for continuous space–time $\times$ discrete space.

6. Conclusions

As described in the introduction and in accordance with several ideas expressed by A. Connes we have explicitly shown how an action for gravity can be obtained in the framework of non-commutative geometry via the Wodzicki residue. In particular, we proved that $\mathrm{Res}(D^{-n+2})$ picks out the second coefficient of the heat kernel expansion of D^2, where D is a Dirac operator on an n dimensional compact Riemannian manifold with $n \geq 4$ and n even. In this article we applied our result to a conventional geometric set up, where we could check that this procedure leads to usual gravity, i.e. it leads to the Einstein–Hilbert action. We also considered an extension of commutative geometry to non-commutative geometry given by the tensor product of the algebra of smooth functions on a manifold and an finite dimensional matrix algebra. In this case we obtained gravity with a cosmological constant.

A natural further question is how to couple Yang–Mills fields to gravity. As we have seen, a Yang–Mills connection which can be part of the Dirac operator does not contribute to the Wodzicki residue that leads to an action of gravity. A solution to this problem is given by adding the Yang–Mills action given by Eq. (3.5) to the gravity action Eq. (5.1).

However, there are some limitations to our approach. The first, common to all models in non-commutative geometry which use Dixmier trace or Wodzicki residue, is the fact that so far we can only describe Riemannian geometry but not pseudo-Riemannian space–time. Furthermore we have proved the theorem in Section 4 only in even dimensions. We hope to come back to these problems in a future publication.

Note: An independent calculation of the Wodzicki residue in 4 dimensions, leading to the same result, was simultaneously done by D. Kastler [14].

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